## Some familiar topics in Algebra & Algebraic Geometry

A student can do a research project in any of the areas listed below. The list below is not exhaustive; students may contact relevant faculty members based on their interests. This may involve lectures in the second year, or a reading course with discussion, as advised by the faculty members.

**Representation Theory of Finite Groups**:

Definition of a group representation and sub-representation, examples. Irreducible representations and tensor product of two representations. Associated characters, Schur's lemma, Orthogonality relations, Decomposition theorems. Induced representations, Compact groups and their representations, Concrete examples. Group algebras, modules, complete reducibility, Wedderburn's theorem. Restriction and induction, Mackey's criterion, Representations of super solvable groups. Artin's Theorem, Brauer's theorem, Applications, Introduction to rationality questions with examples.

Textbooks:


1. G. James, and M. Liebeck, *Representations and characters of groups*, Cambridge Math. Textbooks, Cambridge University Press, Cambridge, 1993.
2. W. Fulton, and J. Harris, *Representation Theory: A First Course*, Graduate texts in mathematics. Vol. 129. New York, NY: Springer, 1991.
3. J.-P. Serre, *Linear Representations of Finite Groups*, Graduate texts in mathematics. Vol. 42. New York, NY: Springer-Verlag, 1977.
4. B. Simons, *Representations of finite and compact groups*, Graduate Studies in Mathematics, 10. American Mathematical Society, Providence, RI, 1996. xii+266 pp.


**Combinatorial Group Theory**:

Word combinatorics - Construction of a free group on a given set, Definition as a universal object. Group as a homomorphic image of a free group, Construction of a group using generators and relations. Elementary properties of free groups, Tietze transformations and Nielsen's method. Automorphisms of free groups, Cayley graph. Verbal subgroup, Presentation of a subgroup (The Reidemeister-Schreier method), HNN extensions. Free products and amalgamation, Embedding theorems, One-relator groups. Small cancellation theory, Finite quotients (Burnside's problem). Dehn's fundamental problems and recent developments.

Textbooks:

1. G. Baumslag, *Topics in combinatorial group theory*, Lectures in Mathematics ETH Zurich, Birkhauser Verlag, Basel, 1993.
2. R. C. Lyndon and P. E. Schupp, *Combinatorial Group Theory*, Reprint of the 1977 edition, Springer, 2001.
3. W. Magnus, A. Karrass and D. Solitar, *Combinatorial group theory, Presentations of groups in terms of generators and relations*, Reprint of the 1976 second edition, Dover Publications, 2004.
4. J-P. Serre, *Trees*, Translated from the French by John Stillwell. Springer-Verlag, Berlin-New York, 1980.

**Lie Algebras**:

Lie Groups, examples of Lie groups, Lie algebra of a Lie group, exponential map, closed linear groups and corresponding Lie algebras. Linear Lie algebras, examples, $A_l$, $B_l$, $C_l$, $D_l$, Abstract Lie algebras.

Ideals, Homomorphisms and representations of Lie algebras, Automorphisms, Solvable and nilpotent Lie algebras, Engel's theorem.

Lie's theorem for solvable Lie algebras, Jordan-Chevalley decomposition, Cartan's criterion, Killing form, Semisimple Lie algebras, Inner derivations, Abstract Jordan decomposition, Complete reducibility of representations.

Casimir element of a representation, Weyl's theorem, Preservation of Jordan decomposition, Representations of $\mathfrak{sl}(2, F)$, Root space decomposition for a semisimple Lie algebra, centralizer of maximal toral subalgebra, Orthogonality properties, Integrality and rationality properties.

Textbooks:

1. James E. Humphreys, *Introduction to Lie algebras and representation theory*, Graduate Texts in Mathematics, Vol. 9. Springer-Verlag, New York-Berlin, 1972.
2. Anthony W. Knapp, *Lie groups, Lie algebras, and cohomology*, Math. Notes, 34 Princeton University Press, Princeton, NJ, 1988.
3. J.P. Serre, *Lie algebras and Lie groups*, 1964 lectures given at Harvard University. Second edition Lecture Notes in Math., 1500 Springer-Verlag, Berlin, 1992.

4. Karin Erdmann and Mark J. Wildon, *Introduction to Lie algebras*, Springer Undergrad. Math. Ser. Springer-Verlag London, Ltd., London, 2006.

**Introduction to Algebraic Geometry**:
Affine algebraic sets, Hilbert Nullstellensatz, Function field, Krull dimension. Graded rings, (quasi-)projective Varieties, rational functions. Algebraic plane curves, morphism, product, finite maps. Normality, Noether normalisation, Birational morphism Tangent spaces, non-singularity, resolution of singularities, blowing up points. Divisors, Bezut's theorem on curves, Class group Intersection numbers, Riemann-Roch theorem.

Textbooks:

1. I. Shafarevich, *Basic Algebraic Geometry I*, (Second Edition) Springer-Verlag (1988).
2. M. Reid, *Undergraduate Algebraic Geometry*, Cambridge University Press (1988).
3. W. Fulton, *Algebraic Curves*, Addison-Wesley Publishing Company, Advanced Book Program (1969).
4. U. Görtz, T. Wedhorn, *Algebraic Geometry I*, Vieweg+Teubner Verlag (2010).

**Algebraic Varieties**: Spaces with sheaves of functions, affine algebraic varieties over an algebraically closed field, the category of algebraic varieties, subvarieties, products, projective varieties, separation, normality, dimension, rational maps, tangent spaces, smoothness, completeness, finite morphisms, constructible sets, divisors, curves, and the Riemann-Roch theorem.

Textbooks:

1. G. R. Kempf, Algebraic Varieties.
2. D. Eisenbud, Commutative Algebra; With a view toward Algebraic Geometry.
3. J. S. Milne, Algebraic Geometry, http://www.jmilne.org/math/.

**Commutative Algebra**: Modules and tensor products, prime ideals, the Zariski topology, rings and modules of fractions, flatness, valuation theory, integral extensions, discrete valuation rings, Dedekind domains, Artinian and Noetherian rings and modules, the Hilbert basis theorem, primary decomposition, Noether normalization, Hilbert's Nullstellensatz, completions, the Krull dimension

Textbooks:

1. S. Bosch, Algebraic Geometry and Commutative Algebra.
2. D. Eisenbud, Commutative Algebra; With a view toward Algebraic Geometry.

**Advanced Complex Analysis**:

Basic properties of holomorphic functions; relations with the fundamental group and covering spaces; the open mapping theorem; the maximum modulus theorem; zeros of holomorphic functions; classification of singularities; meromorphic functions; the Weierstrass factorization theorem; Riemann mapping theorem; the Little Picard theorem. Montel's theorem, Cauchy's integral formula, homotopy form of Cauchy's theorem, argument principle and Hurwitz's theorem. Jensen's formula, entire functions of finite order, Weierstrass infinite products, Hadamard's factorisation theorem, Phrageman-Lindelof theorem Runge's approximation theorem, Mittag Leffler theorem, cohomology form of Cauchy's theorem, Ahlfors version of Schwarz Lemma, Big Picard's theorem, Basic properties of holomorphic/ meromorphic functions on Riemann surface (principle of analytic continuation, open mapping theorem, maximum principle, Weierstrass theorem, Montel's theorem, Riemann extension theorem), Compact Riemann surface associated to irreducible algebraic polynomial in two variables, Finitess of first cohomology group associated to compact Riemann surface and existence of non-constant meromorphic function on a Compact Riemann surface. Riemann-Roch theorem and applications.

<u>Textbooks</u>:

1. R. Narasimhan and Y. Nievergelt, Complex Analysis in one variable, Second edition.

**Riemann Surfaces**:

Definition of complex atlases, complex charts, Riemann surfaces and complex structure on a manifold. Examples of Riemann surfaces; projective line; complex torus; smooth projective plane curves. Holomorphic functions on a Riemann surface; meromorphic functions on a Riemann surface with examples. Holomorphic maps between Riemann surfaces; automorphism groups; degree of holomorphic maps; Euler number for compact Riemann surfaces and Hurwitz formula. Recall basics of covering maps, fundamental groups, group actions on manifolds and quotients. Finite group actions on Riemann surfaces; Hurwitz's theorem; Monodromy of covering and holomorphic maps; monodromy representation. Differential and holomorphic forms; sheaves; vector bundles, line bundles and divisors. Linear equivalence and forms associated to divisors; finiteness theorems for cohomology on a compact Riemann surface; Dolbeault isomorphism; Weyl's lemma on regularity of $\overline{\partial}$ and Serre duality; Rieman-Roch theorem; some applications of the Riemann-Roch theorem; Able-Jacobi map and Abel's theorem.

Textbooks:

1. Rick Miranda, *Algebraic curves and Riemann surfaces*, Grad. Stud. Math., 5 American Mathematical Society, Providence, RI, 1995.
2. Raghavan Narasimhan, *Compact Riemann surfaces*, Lectures Math. ETH Zürich Birkhäuser Verlag, Basel, 1992.
3. M. S. Narasimhan, R. R. Simha Raghavan Narasimhan, C. S. Seshadri, *Riemann Surfaces*, TIFR Pamphlet. Otto Forster, *Lectures on Riemann surfaces*, Translated from the German by Bruce Gilligan Graduate Texts in Mathematics, 81, Springer-Verlag, New York-Berlin, 1981.

**Advanced Algebraic Geometry (Scheme theory)**:
(Pre-)sheaf, Sheaves of modules over topological spaces Spec of a ring, stalk, co-ordinate ring, dimension Scheme, Gluing, reduced and integral scheme Product, separated and proper scheme Projective morphism, flat morphism, birational map Coherent sheaves and vector bundles, divisors and line bundles Sheaf cohomology, derived push-forward and pull-back, base change Serre duality theorem, semicontinuity theorems functor of points, Group scheme, Hilbert scheme.

Textbooks:

1. I. Shafarevich, *Basic Algebraic Geometry II*, (Second Edition) Springer-Verlag (1988).
2. R. Hartshorne, *Algebraic Geometry*, Graduate texts in Mathematics (1977).
3. D. Mumford and T. Oda, *Algebraic Geometry II*, TRIM publications 73 (2015).
4. Q. Liu, *Algebraic Geometry and Arithmetic Curves*, OUP Oxford (2002).
5. D Eisenbud, J. Harris, *The Geometry of Schemes*, Graduate texts in Mathematics (2000),
6. U. Görtz, T. Wedhorn, *Algebraic Geometry I*, Vieweg+Teubner Verlag (2010).

**Homological Algebra**:

Categories; Limits and colimits; Functors and natural transformations; Adjoint functors.

Additive categories and Abelian categories; Chain complexes in an Abelian category; Chain homotopy; Mapping cone and mapping cylinders. Delta functors; Exact functors.

Injective resolutions; Projective resolutions; Cartan-Eilenberg resolutions; Truncation of complexes; EXT and TOR; Double complex and totalization; Spectral sequences and convergence; Leray-Serre spectral sequence; Spectral sequence associated to a filtration and double complex; Grothendieck spectral sequences; spectral sequence associated to an exact couple.

Simplicial objects in a category; Simplicial homotopy groups; Dold-Kan correspondence; Eilenberg-Zilber theorem.

Homotopy category of chain complexes; Triangulated categories; Localization; Derived categories; Left and right exact functors; Derived functors.

Textbooks:


1. Charles A. Weibel, *An introduction to Homological Algebra*, Cambridge Stud. Adv. Math., 38 Cambridge University Press, Cambridge, 1994. xiv+450 pp.
2. H. Cartan and S. Eilenberg, *Homological algebra*, Princeton University Press, Princeton, N. J., 1956.
3. J. J. Rotman, *An introduction to Homological Algebra*, Second edition Universitext Springer, New York, 2009.
4. S. I. Gelfand and Y. I. Manin, *Methods of Homological Algebra*, Second edition Springer Monogr. Math. Springer-Verlag, Berlin, 2003.
5. P.J. Hilton and U. Stammbach, *A course in Homological Algebra*, Graduate Texts in Mathematics, Vol. 4. Springer-Verlag, New York-Berlin, 1971.